Exercises in Fall 2019 PDE course

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1 General questions

Question 1.1 (Giraud). *Let Ω be a bounded open set of $\mathbb{R}^n$ and let $f(x,y)$ and $g(x,y)$ be continuous functions on $\Omega \times \Omega \setminus \{x=y\}$, which satisfy*

$$|f(x,y)| \leq C|x-y|^{\alpha-n} \text{ and } |g(x,y)| \leq C|x-y|^{\beta-n},$$

where $0<\alpha,\beta<n$. Then

$$h(x,y)=\int_\Omega f(x,z)g(z,y)\mathrm{d}z$$

is continuous in $\Omega \times \Omega \setminus \{x=y\}$ and

(a) $|h(x,y)| \leq C|x-y|^{\alpha+\beta-n}$ *if* $\alpha+\beta<n$*;*

(b) $|h(x,y)| \leq C(1+|\log|x-y||)$ *if* $\alpha+\beta=n$*;*

(c) $|h(x,y)| \leq C$ *if* $\alpha+\beta>n$*.*

Hint. *Use the decomposition of Ω: $B_\rho(x)\cap\Omega$, $[B_{3\rho}(y)\setminus B_\rho(x)]\cap\Omega$ and $\Omega\setminus B_{3\rho}(y)$, with $2\rho=|x-y|$.*

Question 1.2. *Let $0<s<1$ and $\nu\geq 0$, then*

$$\int_{\mathbb{R}^n}\frac{\mathrm{d}y}{|x-y|^{n-2s}(1+|y|)^{\nu+4s}} \leq C\frac{\log(2+|x|)}{(1+|x|)^{\min\{n-2s,\nu+2s\}}}.$$

Question 1.3. *Let A,B be real symmetric $n\times n$ matrices with eigenvalues $\{\lambda_i\}$ and $\{\mu_i\}$ satisfying:*

$$\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq 0; \quad \mu_1\leq\mu_2\leq\cdots\leq\mu_n,$$

then $\mathrm{tr}(AB)\geq\sum_{i=1}^n\lambda_i\mu_i$.

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Question 1.4 (Hadamard identity)**.** *Let Ω be a domain in $\mathbb{R}^n$ and $f \in C^2(\bar{\Omega}; \mathbb{R}^n)$, denote by B_{ij} the algebraic cofactor of the element $\frac{\partial f^i}{\partial x^j}, 1 \leq i, j \leq n$ in the determent $J_f(x)$. Then there holds*

$$\sum_{i=1}^{n} \tfrac{\partial}{\partial x^j} B_{ij}(x) = 0, \quad \text{for each } \; i = 1, \cdots, n.$$

*(**Notation.** i: row index; j: column index)*

2 The wave equation

Question 2.1. *Let $\Omega = \{(x, y) \in \mathbb{R}^2; 0 < x < a, 0 < y < b\}$, and use separation of variables to solve the initial boundary value problem*

$$\begin{cases} \partial_t^2 u = \partial_x^2 u + \partial_y^2 u & \text{in } \; \Omega \times (0, \infty); \\ u(x, y, t) = 0 & \text{on } \; \partial\Omega \times (0, \infty); \\ u(x, y, 0) = \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b}, \;\; \partial_t u(x, y, 0) = 0 \quad \text{in } \; \Omega. \end{cases}$$

Question 2.2. *Let $B_1^+ = \{x = (x', x_n) \in \mathbb{R}^n; |x| < 1, x_n > 0\}$ and $\lambda \in \mathbb{R}$, solve the following eigenvalue problem:*

$$\begin{cases} -\Delta u = \lambda u & \text{in } \; B_1^+; \\ u = 0 & \text{on } \; \partial^+ B_1^+; \\ \frac{\partial u}{\partial x_n} = 0 & \text{on } \; \partial' B_1^+. \end{cases}$$

Question 2.3. *Find a formula for the solution $v(x, t) = v(x_1, x_2, t)$ of the Cauchy problem for the two dimensional Klein-Gordon equation:*

$$\begin{cases} \partial_t^2 v = a^2 \Delta v - m^2 a^2 v & \text{for } \; x \in \mathbb{R}^2, t > 0; \\ v(x, 0) = \varphi(x), \;\; \partial_t v(x, 0) = \psi(x), & \text{for } \; x \in \mathbb{R}^2; \end{cases}$$

where a, m are two positive constants.

Question 2.4. *If $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$ and $n \geq 1$, then the solution of*

$$\begin{cases} \partial_t^2 u = \Delta u & \text{for } \; x \in \mathbb{R}^n, t > 0; \\ u(x, 0) = \varphi(x), \;\; \partial_t u(x, 0) = \psi(x), & \text{for } \; x \in \mathbb{R}^n; \end{cases}$$

has the following estimate: there exists a positive constant C such that

$$|u(x, t)| \leq \frac{C}{t^{\frac{n-1}{2}}} \quad \text{for any } \; x \in \mathbb{R}^n, t > 0.$$

If $n=3$ and let $\xi = t+r, \eta = t-r$, where $r=|x|$, then

$$|\partial u| \le ct^{-1}, \quad |\partial_\eta u| \le ct^{-1}, \quad |\partial_\xi u| \le Ct^{-2} \quad \text{as } t\to\infty$$

and

$$|\partial u| \le \frac{C}{(1+r)(t-r)^{\frac{3}{2}}} \quad \text{near } t=r.$$

3 The heat equation

Question 3.1. *Formally check that*

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} \frac{\mathrm{d}^k}{\mathrm{d}t^k} e^{-\frac{1}{t^2}}$$

satisfies $\partial_t u = \partial_x^2 u$ for $x\in\mathbb{R}, t>0$, and $u(x,0)=0$ for $x\in\mathbb{R}$. In other words, we need to prove the convergence of the infinite series.

Question 3.2. *Apply the similarity method to the porous medium equation $\partial_t u = \Delta u^\gamma$ for $x\in\mathbb{R}^n, t>0$, where $\gamma>1$, to obtain the* ***Barenblatt****'s solution:*

$$u_\gamma(x,t) = t^\alpha \left(C - \frac{(\gamma-1)\beta|x|^2}{2\gamma t^{2\beta}} \right)^{\frac{1}{\gamma-1}},$$

where C is a positive constant, $\beta = (n(\gamma-1)+2)^{-1}$ and $\alpha = n\beta$. Indeed, the above solution is also true for $0<\gamma<1$ and $\gamma \neq \frac{n-2}{n}$, so, how about $\gamma = \frac{n-2}{n}$?

Question 3.3. *Find a formula for the solution of the Cauchy problem*

$$\begin{cases} \partial_t u = \Delta u - u, & x\in\mathbb{R}^n, t>0; \\ u(x,0) = \varphi(x), & x\in\mathbb{R}^n; \end{cases}$$

where φ is continuous and bounded. Is the solution bounded? Is it the only bounded solution? Furthermore, let $\lambda\in\mathbb{R}$, can you write down an explicit formula of the solution to

$$\begin{cases} \partial_t u - \Delta u + \lambda u = f(x,t), & x\in\mathbb{R}^n, t>0; \\ u(x,0) = \varphi(x), & x\in\mathbb{R}^n. \end{cases}$$

4 The Laplace equation

Question 4.1. *Let $A(\varepsilon,0)=\{x\in\mathbb{R}^n; 0<\varepsilon<|x|<1\}$ be an annulus in $\mathbb{R}^n, n\geq 2$. Using the method of separation of variables to solve the following Steklov eigenvalue problem in $A(\varepsilon,0)$:*

$$\begin{cases}\Delta u=0 & \text{in } A(\varepsilon,0);\\ \frac{\partial u}{\partial r}=\sigma u & \text{on } S_1;\\ -\frac{\partial u}{\partial r}=\sigma u & \text{on } S_\varepsilon;\end{cases}$$

where $\sigma\in\mathbb{R}_+$ are the Steklov eigenvalues.

Question 4.2. *Let $A(r,0)=\{x\in\mathbb{R}^n; 0<\varepsilon<|x|<1\}$ be an annulus in $\mathbb{R}^n, n\geq 3$. Find a solution to*

$$\begin{cases}\Delta u=0 \quad \text{in } A(\varepsilon,0);\\ u(x)=a, \quad x\in S_\varepsilon; \quad u(x)=1, \quad x\in S_1;\end{cases}$$

where $a\in\mathbb{R}$ and $a>1$. Let $v=u|_{\partial A(\varepsilon,0)}$, find an optimal constant $\varepsilon_0\in(0,1)$ such that

$$\frac{\left(\int_{A(\varepsilon,0)}|u(x)|^{\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{1}{n}}}{\left(\int_{\partial A(\varepsilon,0)}|v(x)|^{\frac{2(n-1)}{n-2}}\mathrm{d}\sigma\right)^{\frac{1}{n-1}}}>\frac{(\mathrm{Vol}(B_1))^{\frac{1}{n}}}{(\mathrm{Vol}(\partial B_1))^{\frac{1}{n-1}}} \tag{4.1}$$

for all $0<r<\varepsilon_0$?[Open problem.]
Hint. *First, it is impossible that the inequality (4.1) holds for all $0<r<1$, i.e. $\varepsilon_0=1$. Next, one of possible new ways is to choose some suitable boundary data, and solve the harmonic function by separation of variables. How about using some Steklov eigenfunction in Question 4.1 or its linear combination with the above test function? Keep trying it.*

Question 4.3. *Let $\Omega=\{(x,y)\in\mathbb{R}^2; 0<x<a, 0<y<b\}$. Solve the following eigenvalue problem by separation of variables:*

$$\begin{cases}\partial_x^2u+\partial_y^2u+\lambda u=0, \quad 0<x<a, 0<y<b;\\ u(0,y)=u(a,y)=0;\\ u(x,0)=u(x,b)=0.\end{cases}$$

Question 4.4. *Show that the bounded solution of the Dirichlet problem of Poisson equation in a half-space is unique. Give unbounded counterexamples.*

Question 4.5. *Let u be a harmonic function in $\mathbb{R}^n$, $n\geq 3$, then the frequency function*

$$\mathcal{F}_u(r):=\frac{r^{2-n}\int_{B_r}|\nabla u|^2\mathrm{d}x}{r^{1-n}\int_{\partial B_r}u^2\mathrm{d}\sigma}$$

is non-decreasing in $(0,\infty)$.

Question 4.6. *Let u be a harmonic function in $\mathbb{R}^n$, $n \geq 3$, then there holds*

$$\int_{\partial B_1(0)} u(R_1 x)u(R_2 x)\mathrm{d}\sigma = \int_{\partial B_1(0)} u^2(\sqrt{R_1 R_2}x)\mathrm{d}\sigma$$

for $R_1, R_2 > 0$. Hint: Use Poisson formula on the ball and symmetry of the ball.

Question 4.7 (The unique continuation for harmonic functions)**.** *Let Ω be an open connected subset of $\mathbb{R}^n, n \geq 3$. Suppose that $u \in C^2(G)$ and $\Delta u = 0$ in G, and $u = 0$ in an open set $\Omega \subset G$, then $u = 0$ in G.*

Question 4.8 (Bôcher)**.** *A positive harmonic function u in a punctured ball $B_1(0)\backslash\{0\}$ must be of the form*

$$\begin{cases} -a\log|x| + h(x) & \text{if } \ n = 2, \\ a|x|^{2-n} + h(x) & \text{if } \ n \geq 3, \end{cases}$$

where a is a nonnegative constant and $h(x)$ is a harmonic function in B_1.

Question 4.9. *A harmonic function in a punctured ball $\overline{B_1}(0) \setminus \{0\}$ which is bounded in $\overline{B_1}(0)$, must be smooth in $\overline{B_1}(0)$.*

Question 4.10. *For a multi-index α with $|\alpha| = 2$, let P be a homogeneous harmonic polynomial of degree 2 with $D^\alpha P \neq 0$. Choose $\eta \in C_c^\infty([-2, 2])$ with $\eta = 1$ when $|x| < 1$, set $t_k = 2^k$, and let $c_k \to 0$ as $k \to \infty$, with $\sum_{k=0}^\infty c_k$ divergent. Define*

$$f(x) = \sum_{k=0}^{\infty} c_k \Delta(\eta P)(t_k x).$$

Show that f is continuous but that $\Delta u = f$ does not have a C^2 solution in any neighborhood of the origin.

5 Open problems

Question 5.1. *Is it possible to find a special solution of the following PDE (one may assume the radial symmetry of the solution and solves the corresponding ODE):*

$$\Delta^2 u^{\frac{n-4}{n+4}} - \rho_1 u - \rho_2 x \cdot \nabla u = 0, \ \ u > 0 \ \text{ in } \ \mathbb{R}^n. \tag{5.1}$$

where $n \geq 5$ and ρ_1, ρ_2 are constant. Furthermore, is it possible to find a special solution to

$$\partial_t u + \Delta^2 u^\gamma = 0 \quad \text{for } \ (x,t) \in \mathbb{R}^n \times (0, T),$$

where $\gamma > 0, T > 0$ and $n \geq 5$.

Question 5.2. *Let $k \in \mathbb{N}, \alpha \in \mathbb{R}_+$ and set $Q = 1 + k(\alpha + 1)$, to solve the following ODE of $\Phi = \Phi(t)$:*

$$t(t+1)\Phi'' + \frac{k+1}{2}(2t+1)\Phi' + \frac{Q(Q-2)}{4(\alpha+1)^2}\Phi + \frac{1}{(\alpha+1)^2}\Phi^{\frac{Q+2}{Q-2}} = 0.$$